## **5. Oscillation of the string with fixed ends**

We consider the movement of the bounded string with fixed ends. This phenomenon is described by the first boundary problem for the vibrating string equation. Using the method of separation of variables, we transform the partial differential equation to two ordinary differential equations, which are connected by a common constant. The spatial equation with boundary conditions is called the Sturm–Liouville problem. This problem has an infinite set of solutions. Using Fourier series properties and the given initial condition, determine the solution of the initial problem as a Fourier series. The oscillation of the string is obtained as an application of these results.

### **5.1. Problem statement**

Consider the movement of the bounded string. We have the vibrating string equation

 *utt = a2 uxx*, 0 < *x* < *L*, *t* > 0, (5.1)

where *L* is the length of the string. Suppose the ends of the string are fixed. Then we have the boundary conditions

 *u*(0,*t*) = 0, *u*(*L*,*t*) = 0, *t* > 0. (5.2)

The initial state *ϕ* =*ϕ*(*x*) and the initial velocity*ψ* =*ψ*(*x*) of the string are given. Then we have the initial conditions

 *u*(*x*,0) = *ϕ*(*x*), *ut*(*x*,0) = *ψ*(*x*), 0 < *x* < *L*. (5.3)

The system (5.1) – (5.3) is called the ***first boundary problem*** for the vibrating string equation, and the equalities (5.2) are called the ***first order boundary conditions***.

### **5.2. Method of separation of variables**

Try to find the solution of the equation (5.1) as a product of the functions of one variables

 *u*(*x*,*t*) = *X*(*x*) *T*(*t*), (5.4)

where the functions *X* and *T* are unknown. We would like to choose it such that the formula (5.4) gives the solution of the considered problem. This is the basis of the method of separation of variables.

Put the function *u* from the equality (5.4) to the vibrating string equation (5.1). We get

*X*(*x*) *T''*(*t*) = *a*2 *X''*(*x*) *T*(*t*),

where *T''* and *X''* are second derivatives of the functions *X* and *T* of one variable. Divide this equality by *a*2*XT.* We obtain



We have the equality of the functions of different variables. It can be true only the values at the right hand-side and the left hand-side are constant. Denote this constant by *λ*. We obtain two ordinary differential equations

 *T''*(*t*) = *a*2*λ T*(*t*), *t* > 0, (5.5)

 *X''*(*x*) = *λ X*(*x*), 0 < *x* < *L.*  (5.6)

Thus, our partial differential equation was be transformed to two ordinary differential equations with different independent variable. This is the idea of the method of separation of variables.

Now we put the function *u* from the equality (5.4) to the boundary conditions (5.2). We have

*X*(0) *T*(*t*) = 0, *X*(*L*) *T*(*t*) = 0, *t* > 0.

The product of two values can be zero, if one of the multipliers is zero. If *T*(*t*) = 0 is zero for all time, then the function *u* is zero everywhere because of the formula (5.4). However, this contradicts the initial conditions (5.3). Therefore, we obtain the equalities

 *X*(0) = 0, *X*(*L*) = 0. (5.7)

The second order differential equation (5.6) with boundary conditions (5.7).

Of course, the problem (5.6), (5.7) has the trivial zero solution. However, if *X*(*x*) is zero for all *x*, then the function *u* will be zero too because of the equality (5.4). However, this contradicts again the initial conditions (5.3). The problem of finding non-zero solution of the system (5.6), (5.7) with arbitrary parameter *λ* is called the ***Sturm–Liouville problem***.

Try to analyze this problem. Then we consider the differential equation (5.5) and return to the boundary problem (5.1) – (5.3), using the formula (5.4).

### **5.3. Sturm–Liouville problem**

Find the solution of the problem (5.6), (5.7). We have the linear homogeneous second order differential equation

*X''*(*x*) – *λ X*(*x*) = 0.

Using the standard result of the ordinary differential equation theory, we consider the characteristic equation

*z*2 – *λ* = 0.

Then we determine its solution



The result depends from the sign of the constant *λ*. Thus, it is necessary to consider three different cases.

Suppose the constant *λ* is positive. Then the general solution of the equation (5.6) has the exponent form

  (5.8)

where *c*1 and *c*2 are arbitrary. Using the boundary conditions (5.7), we get





We have the system of two linear algebraic equations with respect to the constant *c*1 and *c*2. Determine *c*1 = 0, *c*2 = 0. Therefore, the value *X*(*x*) is zero because of the equality (5.8). However, we would like to determine a non-zero solution of the problem. Hence, this case is not applicable.

Now suppose the constant *λ* is zero. Then the general solution of the equation (5.6) has the linear form

  (5.9)

Using the boundary conditions (5.7), we obtain





This system has zero solution, and the function *X* is zero too. Therefore, the second is not applicable too.

Finally, we suppose the constant *λ* is negative. Then the general solution of the equation (5.6) has the trigonometric form

  (5.10)

Using the boundary conditions (5.7), we get





By first of these equalities, we have



We conclude that one of the multipliers of the value at the left hand-side is zero. If *c*1 = 0, then *X* is zero function because of the equality (5.10). Thus, we determine



This equality can be true, if



Now we determine the infinite family of parameters

  (5.11)

Thus, there exists the infinite set of non-zero solutions of boundary problem (5.6), (5.7). There are the functions

  (5.12)

We use the constant *ck* here, because for *k* we can have the different constant. Any function *Xk*with arbitrary constant *ck* is the solution of the Sturm–Liouville problem.

Now we return to the vibrating string equation.

### **5.4. Vibrating string equation with first order boundary condition**

Consider the differential equation (5.5) with parameter *λ* is equal to *λk*. Determine the characteristic equation

*z*2 – *a*2*λk* = 0.

Then its general solution for the arbitrary *k* is

  (5.13)

where the constants *ak* and *bk* are arbitrary. Put the values of the functions *Xk* and *Tk* from the equalities (5.12) and (5.13) to the formula (5.4). We find the functions

  (5.14)

where the constants  are arbitrary.

The functions *uk* satisfy the vibrating string equation (5.1) and the boundary conditions (5.2) for all values *k* and the constants  Note the general property of the equation (5.1) (and any linear homogeneous equation too). If we have two solutions of the equations, then its sum satisfies this equation too. Sum all functions *uk*. Using the formula (5.14), we find

  (5.15)

This function the equation (5.1) and the boundary conditions (5.2). Now it necessary to choose the coefficients  such that the function *u* from the equality (5.15) satisfies the initial conditions (5.2) too.

### **5.5. Solution of the problem (5.1) – (5.3).**

Put the function *u* from the equality (5.15) to the first initial condition (5.3). We get



Now differentiate formally the equality (5.15) by *t*. We have



Determine here *t=*0. Using the second initial condition (5.3), we obtain



Thus, we obtain the equalities

  (5.16)

  (5.17)

The relations (5.16), (5.17) give the representations of the functions *ϕ* and *ψ* as ***Fourier series***. Using Fourier series theory, we know that any not very bad function *ϕ* has the representation as Fourier series (5.6) with ***Fourier coefficients***

  (5.18)

Now find the Fourier coefficients of the function *ψ*. We have



Then we determine

  (5.19)

Thus, the solution of the first boundary problem for the vibrating string equation is the function *u* that is determined by the formula (5.15) with Fourier coefficients (5.18), (5.19).

### **5.6. Oscillation of the string with fixed ends**

Consider the partial case of the problem (5.1) – (5.3). Let us analyze the string of the length *L=π* with coefficient *a =* 1. Then we have the vibrating string equation

 *utt = uxx*, 0 < *x* < *π*, *t* > 0. (5.20)

The ends of the string are fixed. Then we have the boundary conditions

 *u*(0,*t*) = 0, *u*(*π*,*t*) = 0, *t* > 0. (5.21)

Suppose the initial state of the string is *ϕ* =sin *x*, and the initial velocity*ψ* is zero. Then we have the initial conditions

 *u*(*x*,0) = sin *x*, *ut*(*x*,0) = 0, 0 < *x* < *L*. (5.22)

Using the formula (5.15), determine the solution of the problem (5.20) – (5.22) by the formula

  (5.23)

Find the coefficients  by the formulas (5.19). We get



Now determine the coefficients  by the formulas (5.20). We have



Using the formula of sinus product, we have

.

Find



Calculate the second integral



The value of the first integral depends from the number *k.* If *k*>1, we find



Then for *k =* 1 we have the integral



Finally, we determine



Put the results to the formula (5.23). Thus, the solution of the problem (5.20) – (5.22) is

  (5.24)

Give the physical interpretation of this result. For any fixed time *t* the string has the form of sinus. However, the position of the string changes with respect to the time. Any fixed point *x* oscillates with period 2*π.* Its swing amplitude depends from the point *x*. This is equal to sin *x*. For example, the middle of the string that is the point *x = π*/2 has the position 1 at the time *t =* 0, √2/2 for *t = π*/4, 0 for *t = π*/2, -√2/2 for *t =* 3*π*/4, -1 for *t = π*, -√2/2 for *t =* 5*π*/4, 0 for *t = π*, √2/2 for
*t =* 7*π*/4, 1 for *t =* 2*π*, etc., see the following figure.12



Figure 5.1. Oscillation of the string.

Suppose now the parameter *a* is arbitrary. Then the solution of the considered problem will be



This has analogical sense as the formula (5.24). However, the frequency of the oscillation depends from *a.* If *a* > 1, the frequency will be greater, and the velocity of the movement will be greater too. If *a* < 1, the frequency will be less, and the velocity of the movement will be less too.

### **Conclusions**

* The movement of the string with fixed ends is described by the first boundary problem for the vibrating string equation.
* The vibrating string equation can be transformed to two ordinary differential equations by the method of separation of variables.
* The spatial ordinary differential equation with boundary conditions, i.e. Sturm–Liouville problem has the infinite set of solutions.
* The vibrating string equation with homogeneous boundary conditions has the infinite set of solutions.
* The solution of the first boundary problem for the vibrating string equation has the representation as a Fourier series.
* The Fourier coefficients of this representation are determined by the initial conditions of the considered problem.
* The oscillation of the string can be analyzed as an application of these results.

### Task. **Oscillation of the string with fixed ends**

Consider first order boundary problem for the vibrating string equation:

*utt = a2 uxx*, 0 < *x* < *L*, *t* > 0,

*u*(0,*t*) = 0, *u*(*L*,*t*) = 0, *t* > 0,

*u*(*x*,0) = *ϕ*(*x*), *ut*(*x*,0) = *ψ*(*x*), 0 < *x* < *L*.

Table of parameters

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variant | *L* | *a* | *ϕ*(*x*) | *ψ*(*x*) |
| 1 | 1 | 2 | - sin π*x* | 0 |
| 2 | π | 1 | 0 | sin *x* |
| 3 | 2 | ½ | sin 2π*x* | 0 |
| 4 | 2π | 2 | sin (*x/*2) | 0 |
| 5 | 1 | 1 | 0 | sin π*x* |
| 6 | π | ½ | sin *x* | 0 |
| 7 | 2 | 2 | 0 | sin 2π*x* |
| 8 | 2π | ½ | 0 | - sin (*x/*2) |

It is necessary to find the solution of the problem, to show the graph (position of the string for the different time points), and to give the physical interpretation of the results.